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Solution to Laplace's Equation Using Quantum Calculus

Pintu Bhattacharya[1], Ravi Ranjan[2]
[1]*Assistant Professor, Department of Physics, L. N. College, B. R. A. Bihar University, Muzaffarpur, Bihar, India*
[2]*Assistant Professor, Department of Physics, Nitishwar College, B. R. A. Bihar University, Muzaffarpur, Bihar,India*
Corresponding Author Orcid ID: 0009-0007-9801-4484

ABSTRACT
The quantum calculus emerged as a new type of unconventional calculus relevant to both mathematics and physics. The study of quantum calculus or q-calculus has three hundred years of history of development since the era of Euler and Bernoulli, and was appeared as one of the most arduous techniques to use it in mathematics as well as physical science. At present, it is used in diverged mathematical areas like number theory, orthogonal polynomials, basic hypergeometric functions, etc. Furthermore, in order to get analytical approximate solutions to the ordinary as well as partial differential equations, q-reduced differential technique and quantum separation of variable technique are used in mathematics, Mechanics, and physics. In this paper, Laplace's equation, a well-known equation in both Physical and Mathematical sciences, has been solved extensively based on the basics of $q-$calculus, $q-$transformation methods, and q-separation of variable method. In addition, solutions to the Laplace's equation as obtained by using different boundary conditions are revisited and reviewed. Consequently, all the necessary basics of q-calculus are displayed one by one, and thereafter, the process of finding its solution in view of quantum calculus is described extensively. In order to find out the exact solutions the dimensionality of all the parameters related to the problem has been described. As an essential outcome, it is also found that, as q tends to 1, the solution takes the form as it is in general physics. Hence, this article presents a review and extension that describe the solution to Laplace's equation in view of both Leibnitz and quantum calculus. Thus, it can add a pedagogical exercise for the students of both physical and mathematical sciences to understand the usefulness of quantum calculus.
Keywords— quantum calculus, q-calculus, Laplace's equation, q-transformation method, q-separation of variable method.

1. Introduction

Quantum calculus is basically a different type of calculus provided a different perspective to solve problems in science and engineering. It is a non-Newtonian calculus without limit and was introduced a long back by L. Euler (1707-1783) and Carl G. Jacobi (1804-1851). The $q-$calculus deals with unconventional calculus and reduces to the classical calculus when uses limit and gives a different analytical ambience. In addition, 'q-calculus', being the calculus of finite differences, is more straightforward, systematic and transparent. Though, in the first decade of nineteenth century, Jackson studied it rigorously [1-3], it attained more attention due to its name after the emergence of quantum mechanics by Albert Einstein in 1905. Jackson developed the q−derivative and q−integral in a systematic way different from Leibnitz approach and hence, geometrical interpretation of the q−calculus has been recognized through studies on quantum groups [4].
In recent decades, a number of problems has been studied in different fields of mathematics, physics, statistics, and engineering using q-calculus [5-31]. As for example, a subsequent development in q-derivative and q-integral can also be useful to analyze physical systems in a different way as mentioned and described in Ref [9-12], partial q-difference equation is studied by

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different methods like separation of variables, the techniques of Lie symmetry, and q-integral transforms [16-18]. In addition, q-calculus is applicable in the areas of ordinary fractional calculus, basic hypergeometric functions, the theory of relativity, q−difference and q−integral equations and in q−transform analysis [12, 18-23].
Nowadays, $q-$calculus makes a bridge between mechanics and physics. Occasionally, maximum scientists use q−calculus as a mathematical model are physicist and are using it in several modern field of research like 'quantum Field Theory, Newtonian quantum gravity, Special and General relativity, Molecular and nuclear spectroscopy, and even in String theory [4, 32,33].
However, quantum calculus is popular today and may be considered as an extension of classical calculus discovered by Issac Newton (1643-1727) and G. W. Leibniz (1646- 1716). In the latter para, we shall discuss the primordial fact for its nomenclature. In fact, in q-calculus, the $h-$operator, and $q-$operator, are defined by,

$$D_h f(z) = \frac{f(z+h) - f(z)}{h} \qquad \dots (1)$$

And,

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \qquad \dots (2)$$

for real or complex $z\, and\, h > 0$. It gives us the exciting world of h−calculus and q−calculus, respectively. Here h stands for Planck's constant and q stands for quantum calculus and the well-known relation is $q = e^{ih} = e^{2\pi i\hbar}$.
The current work is mainly motivated by a recent series of works [4,29,34-36]. This paper basically, expands the boundaries of previous works by solving the Laplace's equation with the help of different methods in quantum calculus.

2. Quantum calculus preliminaries

Let us first summarize, the essential basics of q- calculus, needed to solve the Laplace's equation as we obtained in Ref. [32, 34-37].

Definition 1: In q-calculus, $[m]_q = \frac{q^m - 1}{q-1}$ for any positive integer m such that $\lim_{q\to 1}[m]_q = m$. Hence, $[0]_q = 0$ and $[1]_q = 1$.

Definition 2: If f(t) is an arbitrary function, its q-differential is $d_q f(t) = f(qt) - f(t)$ and the q-derivative is $D_q f(t) = \frac{d_q f(t)}{d_q t} = \frac{f(qt) - f(t)}{(q-1)t}$. It is also to be noted that $\lim_{q\to 1} D_q f(t) = \frac{df(t)}{dt}$. Thus, $D_q t^m = \frac{(qt)^m - (t)^m}{(q-1)t} = \frac{q^m - 1}{q-1} t^{\,m-1} = [m]_q t^{\,m-1}$.

Definition 3: The Jackson q-integral (definite) is given as
$\int_0^a f(t) d_q t = (1-q) a \sum_{l=o}^{\infty} q^l f(q^l a)$, Hence $\int_0^a D_q f(t) d_q t = f(a) - f(0)$

Definition 4: The quantum factorial $[m]_q!$ is defined as

$$[m]_q! = \begin{matrix} [0]_q! = 1 \\ \prod_{m=1}^{m} [m]_q \end{matrix}$$

Definition 5: Small q-exponential function $e_q(kx)$ is represented as,

$$e_q(kx) = \sum_{l=0}^{\infty} \frac{(kx)^l}{[m]_q!}$$

Definition 6: $\int e_q(kx) d_q x = \frac{1}{k} e_q(kx) + c$ where c is a real constant.

Definition 7: Considering their Euler expression of small $q-$exponential, the q-analogous sine and cosine functions can be expressed as

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$$sin_q(mx) = \frac{e_q^{imx} - e_q^{-imx}}{2i} \text{ and } cos_q(mx) = \frac{e_q^{imx} + e_q^{-imx}}{2}$$

And,

$$sinh_q(mx) = \frac{e_q^{mx} - e_q^{-mx}}{2} \text{ and } cosh_q(mx) = \frac{e_q^{mx} + e_q^{-mx}}{2}$$

Definition 8: The q-derivative of a function $f(x)$ is defined as,

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, 0 < |q| < 1$$

And the partial q −derivative of a function $f(x_1, x, x_3 \ldots x_n)$ with respect to a variable x_i is defined as,

$$D_{q,x_i} f = \frac{\partial_q f}{\partial_q x_i} = \frac{f(x_1, x_2, x_3, \ldots, qx_i, \ldots x_n) - f(x_1, x_2, x_3 \ldots x_n)}{(q-1)x_i}, x \neq 0, 0 < q < 1$$

$$[D_{q,x_i} f]_{x_i=0} = \left[\frac{\partial_q f}{\partial_q x_i}\right]_{x_i=0} = \lim_{z_i \to 0} \frac{\partial_q f}{\partial_q x_i}$$

Definition 9: The second order partial q-derivative of a function $f(x_1, x, x_3 \ldots x_n)$ with respect to a variable x_i is defined as,

$$\frac{\partial_q^2 f}{\partial_q x_i^2} = \frac{1}{(q-1)x_i}\left[\frac{\partial_q f(x_1, x_2, x_3, \ldots, qx_i, \ldots x_n)}{\partial_q x_i} - \frac{\partial_q f(x_1, x_2, x_3 \ldots x_n)}{\partial_q x_i}\right]$$

$$= \left[\frac{f(x_1, \ldots, q^2 x_i, \ldots x_n) - f(x_1, \ldots, qx_i \ldots x_n)}{(q-1)qx_i} - \frac{f(x_1, \ldots, qx_i, \ldots x_n) - f(x_1, \ldots x_n)}{(q-1)x_i}\right] \Big/ (q-1)x_i$$

$$= \left[f(x_1, x_2, x_3 \ldots, q^2 x_i, \ldots x_n) - (q+1) f(x_1, x_2, x_3, \ldots, qx_i, \ldots x_n) + qV_q(x_1, x_2, x_3)\right] \Big/ q(q-1)^2 x_i^2$$

2.1 Reduced q-differential transform method

In order to solve partial differential equation, let us now define transformed function and q-difference inverse transform function as described in [29, 35].

In order to solve partial differential equation, let us now define transformed function and q-difference transform function as essential one. Suppose, all q-differentials of $w(x,t)$ exist in some neighborhood $t = \tau$, the transformed function,

$$W_l(x) = \frac{1}{[l]_q!}\left[\frac{\partial_q^l w(x,t)}{\partial_q t^l}\right]_{t=\tau} \quad \ldots (3)$$

And the q-difference inverse transform of $W_l(x)$ is defined as,

$$w(x,t) = \sum_{l=0}^{\infty} W_l(x)(t-\tau)^l \quad \ldots (4)$$

Thus substituting (3) to (4), we obtain,

$$w(x,t) = \sum_{l=0}^{\infty} \frac{1}{[l]_q!}\left[\frac{\partial_q^l w(x,t)}{\partial_q t^l}\right]_{t=\tau} (t-\tau)^l$$

Now at $\tau = 0$, if, $f(x,t) = \frac{\partial_q w(x,t)}{\partial_q x}$, then from (3)

$$F_l(x) = \frac{1}{[l]_q!}\left[\frac{\partial_q^l f(x,t)}{\partial_q t^l}\right]_{t=0} = \frac{1}{[l]_q!}\left[\frac{\partial_q^l}{\partial_q t^l}\left\{\frac{\partial_q w(x,t)}{\partial_q x}\right\}\right]_{t=0} = \frac{1}{[l]_q!}\left[\frac{\partial_q}{\partial_q x}\left\{\frac{\partial_q^l}{\partial_q t^l} w(x,t)\right\}\right]_{t=0}$$

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$$F_l(x) = \frac{\partial_q}{\partial_q x}\left[\frac{1}{[l]_q!}\left\{\frac{\partial_q^l}{\partial_q t^l} w(x,t)\right\}\right]_{t=0} = \frac{\partial_q}{\partial_q x} W_l(x)$$

... (5)

Again, $\tau = 0$, if, $f(x,t) = \frac{\partial_q^k w(x,t)}{\partial_q x^k}$, then from (3),

$$F_l(x) = \frac{1}{[l]_q!}\left[\frac{\partial_q^l w(x,t)}{\partial_q t^l}\right]_{t=0} = \frac{1}{[l]_q!}\left[\frac{\partial_q^l}{\partial_q t^l}\left\{\frac{\partial_q^k w(x,t)}{\partial_q x^k}\right\}\right]_{t=0} = \frac{1}{[l]_q!}\left[\frac{\partial_q^{l+k}}{\partial_q t^{l+k}} w(x,t)\right]_{t=0}$$

$$= \frac{[l+k]_q!}{[l]_q!}\left[\frac{1}{[l+k]_q!}\frac{\partial_q^{l+k}}{\partial_q t^{l+k}} w(x,t)\right]_{t=0} = \frac{[l+k]_q!}{[l]_q!} W_{l+m}(x)$$

Thus,

$$F_l(x) = [l+1]_q [l+2]_q [l+3]_q \ldots [l+k]_q W_{l+m}(x)$$

... (6)

3. Solution to Laplace's Equation in quantum calculus

Laplace's equation is one of the ubiquitous equations in mathematical and physical science. Specifically, to solve boundary value problems in electrostatics, it gives unique solution under suitable boundary conditions and the solution must satisfy superposition principle. It can be formulated in different coordinate system depending on the involvement of the geometry. However, for convenience, we consider the 3-D cartesian coordinate for simplicity and the equation may be written as,

$$\Delta V = \nabla^2 V = \sum_{i=1}^{3} V_{x_i x_i} = \sum_{i=1}^{3} \frac{\partial^2 V}{\partial x_i^2} = 0 \qquad \ldots (7)$$

The general solution can be obtained by separation of variable method considering the solution as described in [39, 40]

$$V(x_1, x_2, x_3) = \prod_{i=1}^{3} X_i(x_i) \qquad \ldots (8)$$

And the factorized solution can be written as,

$$V_{k_1,k_2,k_3}(x_1,x_2,x_3) = \begin{Bmatrix} cos(k_1x_1) \\ sin(k_1x_1) \end{Bmatrix} \begin{Bmatrix} cos(k_2x_2) \\ sin(k_2x_2) \end{Bmatrix} \begin{Bmatrix} \exp\,(k_3x_3) \\ \exp\,(-k_3x_3) \end{Bmatrix} \qquad \ldots (9)$$

Where, x_1, x_2, x_3 are the three cartesian co-ordinates, and k_1, k_2, k_3 are constant to be determined from particular boundary conditions and $k_3 = \left(k_1^2 + k_2^2\right)^{1/2}$. It is possible to represent the last factor of the right-hand side of (9) in terms of sinh and cosh form. It is also to be noted that, (9) is the basic solution from which we can find out the actual or exact solution to the boundary value problem.

On the other hand, in 2-D, if we consider the 'separation constant' κ^2, Laplace's equation reduces to

$$-\frac{1}{X_1(x_1)}\frac{d^2X_1(x_1)}{dx_1^2} = \frac{1}{X_2(x_2)}\frac{d^2X_2(x_2)}{dx_2^2} = \kappa^2$$

... (10)

Then, we can obtain the factorial solution as mentioned in Table 1. Here in (10), it is observed that the solution can have only one constant (let it be κ) depending the nature of which solutions are to be obtained in different factorized form as mentioned in Table 1.

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Table 1: 2-D solution (cartesian) of Laplace's equation

$\kappa^2 = 0$	$\kappa^2 \leq 0, \kappa \to j\kappa'$	$\kappa^2 \geq o$
x_1	$\begin{Bmatrix} cosh\kappa' x_1 \\ sinh\kappa' x_1 \end{Bmatrix} \begin{Bmatrix} cos\kappa' x_2 \\ sin\kappa' x_2 \end{Bmatrix}$	$\begin{Bmatrix} cos\kappa x_1 \\ sin\kappa x_1 \end{Bmatrix} \begin{Bmatrix} cosh\kappa x_2 \\ sinh\kappa x_2 \end{Bmatrix}$
x_2	$\begin{Bmatrix} cosh\kappa' x_1 \\ sinh\kappa' x_1 \end{Bmatrix} \begin{Bmatrix} sin\kappa' x_2 \\ cos\kappa' x_2 \end{Bmatrix}$	$\begin{Bmatrix} sin\kappa x_1 \\ cos\kappa x_1 \end{Bmatrix} \begin{Bmatrix} cosh\kappa x_2 \\ sinh\kappa x_2 \end{Bmatrix}$
$x_1 x_2$	$\begin{Bmatrix} e^{-\kappa' x_1} \\ e^{\kappa' x_1} \end{Bmatrix} \begin{Bmatrix} sin\kappa' x_2 \\ cos\kappa' x_2 \end{Bmatrix}$	$\begin{Bmatrix} sin\kappa x_1 \\ cos\kappa x_1 \end{Bmatrix} \begin{Bmatrix} e^{-\kappa x_2} \\ e^{\kappa x_2} \end{Bmatrix}$
A constant	$\begin{Bmatrix} e^{-\kappa' x_1} \\ e^{\kappa' x_1} \end{Bmatrix} \begin{Bmatrix} cos\kappa' x_2 \\ sin\kappa' x_2 \end{Bmatrix}$	$\begin{Bmatrix} cos\kappa x_1 \\ sin\kappa' x_1 \end{Bmatrix} \begin{Bmatrix} e^{-\kappa x_2} \\ e^{\kappa x_2} \end{Bmatrix}$

But, if we consider V as an analytic function such that both the real and imaginary part satisfy the 2-D Laplace's equation, then the real part of the solution in polar co-ordinate must be a logarithmic function and can be written as in ref. [39,40],

$$V_{real} = logr \qquad \ldots (11)$$

3.1 Solution to 2-D Laplace's equation using reduced q-differential transform method

The 2-D Laplace's equation in view of quantum calculus can be written as,

$$\frac{\partial_q^2 V_q(x_1, x_2)}{\partial_q {x_2}^2} = -\frac{\partial_q^2 V_q(x_1, x_2)}{\partial_q {x_1}^2} \qquad \ldots (12)$$

In order to solve the equation, let the boundary conditions are

$$V_q(x_1, 0) = f(k_1 x_1), k_1 \neq 0 \qquad \ldots (13)$$

$$\frac{\partial_q V_q(x_1, 0)}{\partial_q x_2} = 0 \qquad \ldots (14)$$

Here, k_1 is constant to be determined from boundary conditions.

Thus, letting, $\tau = 0$, remembering reduced q-differential transform method (as discussed in section 2.1), and rearranging (12), the x-part of the solution could be obtained. And (12) can be rewritten as,

$$[p+1]_q [p+2]_q W_{p+2}(x_1) = -\frac{\partial_q^2 W_p(x_1)}{\partial_q {x_1}^2} \qquad \ldots (15)$$

Now, as $[1]_q = 1$

$$W_0(x_1) = f(k_1 x_1) \qquad \ldots (16)$$

$$W_1(x_1) = 0 \qquad \ldots (17)$$

As the solution to the Laplace's equation will be unique one, we consider the function $f(kx_1)$ to be periodic and single valued then $f^{(n)}(x_1) = k_1^n f(x_1)$

$$W_2(x_1) = \frac{{W_0}''(x_1)}{[1]_q [2]_q} = \frac{f''(k_1 x_1)}{[1]_q [2]_q} = \frac{f^{(2)}(k_1 x_1)}{[2]_q!} = \frac{k_1^2 f(k_1 x_1)}{[2]_q!} = \frac{k_1^2 W_0(x_1)}{[2]_q!}$$

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… (18)

$$W_3(x) = 0$$

… (19)

$$W_4(x) = \frac{k_1^4 W_0(x_1)}{[4]_q!}$$

… (20)

$$w_5(x) = 0$$

… (21)

$$w_6(x) = \frac{k_1^6 W_0(x_1)}{[6]_q!}$$

… (22)

As the solution to the Laplace's equation must obey the superposition theorem, Thus

$$V_q(x_1, x_2) = \sum_{n=0}^{\infty} W_p(x_1)(k_2x_2)^n$$

… (23)

Here, we introduce a second constant k_2 to make k_2x_2 dimensionless. Hence,

$$V_q(x_1, x_2) = W_0(x_1) + k_2x_2W_1(x_1) + k_2^2x_2^2W_2(x_1) + k_2^3x_2^3W_3(x_1) + k_2^4x_2^4W_4(x_1) + k_2^5x_2^5W_5(x_1) + k_2^6x_2^6W_5(x_1)\ldots$$

$$V_q(x_1, x_2) = W_0(x_1)\left[1 + \frac{k_2^2x_2^2}{[2]_q!} + \frac{k_2^4x_2^4}{[4]_q!} + \frac{k_2^6x_2^6}{[6]_q!} + \ldots\right]$$

… (24)

$$V_q(x_1, x_2) = W_0(x_1)sinh_q(k_2x_2)$$

… (25)

Now the solution depends on the choice of $W_0(x_1)$, Let it be, A $e_q^{k_1x_1}$, Then the exact solution becomes,

$$V_q(x_1, x_2) = Ae_q^{k_1x_1}sinh_q(k_2x_2)$$

… (26)

Here, A is a constant to be determined from proper boundary conditions and (26) represents similarity with one of the factorized solutions as displayed in the Table-1. Thus, considering the actuality of the boundary value problem, we can find out different factorized solution using quantum calculus. For example, if we use, $V_q(x_1, 0) = 0$, and $\partial_q V_q(x_1,0)/\partial_q x_2 = (sinpx)/p\,, p \neq 0$,in (13), and (14) respectively, then the solution will be, $\left(sin_q px_1 sinh_q pt\right)/p^2$ see ref. [30].

3.2 Solution to 3-D Laplace's equation using q-separation of variable method

In this section, we will solve 3-D Laplace's equation using q-separation of variable method as discussed in ref. [29]. In view of quantum calculus, three-dimensional Laplace's equation can be written as,

$$\sum_{i=1}^{3} \frac{\partial_q^2 V_q}{\partial_q x_i^2} = 0$$

… (27)

Here, $V_q = V_q(x_1, x_2, x_3)$, remembering 'Definition 9', and substituting $\frac{\partial_q^2 V_q}{\partial_q x_i^2}$, we obtain

$$V_q(q^2x_1,x_2,z)+V_q(x_1,q^2x_2,x_3)+V_q(x_1,x_2,q^2x_3)$$
$$-(q+1)\{V_q(qx_1,x_2,x_3)+V_q(x_1,qx_2,x_3)+V_q(x_1,x_2,qx_3)\}$$
$$+3qV_q(x_1,x_2,x_3)=0$$

… (28)

Introducing the method of separation of variables, considering $V_q(x_1,x_2,x_3)=X_1(x_1)X_2(x_2)X_3(x_3)$ and substituting all the similar forms of $V(q^2x_1,x_2,z),V(q^2x_1,x_2,z),V(x_1,x_2,qx_3)$, we obtain,

$$\sum_{i=1}^{3}\left[\frac{X_i(q^2x_i)}{X_i(x_i)}-(q+1)\frac{X_i(qx_i)}{X_i(x_i)}+3q\right]=0$$

… (29)

In order to solve (28), intuitively we take the resulting function to be logarithmic Thus, we can say, $X_i(x_i)=\zeta^{log_q^{x_i}},X_i(qx_i)=\zeta^{log_q^{qx_i}}=\zeta X_i(x_i), and\ X_i(q^2x_i)=\zeta^2X_i(x_i)$. Hence, after substitution of these $X_i(x_i),X_i(qx_i)$, and $X_i(q^2x_i)$ in (28), we obtain the characteristic equation as,

$$\zeta^2-(q+1)\zeta+q=0$$

… (30)

As, q, and 1 are the two roots of this equation, the quantum solutions of (27) are,

$$V_q=\sum_{i=1}^{3}\beta_i\,q^{log_q^{x_i}}\qquad,\zeta=q$$

… (31)

And,

$$V_q=\sum_{i=1}^{3}\beta_i\,log_q^{x_i}\qquad,\zeta=1$$

… (32)

Where, β_i , are constant to be find out from boundary conditions.

3.3 An alternate approach to solve Laplace's Equation

In last two sections, we describe two different ways to solve 2-D and 3-D Laplace's equation. This equation can also be solved by using a different technique in quantum calculus. In view of quantum calculus, (7), and (10), can be rewritten as,

$$\sum_{i=1}^{3}\frac{\partial_q^2V_q}{\partial_qx_i^2}=0 \qquad … (33)$$

$$-\frac{1}{X_1(x_1)}\frac{d_q^2X_1(x_1)}{dx_1^2}=\frac{1}{X_2(x_2)}\frac{d_q^2X_2(x_2)}{dx_2^2}=\kappa^2$$

… (34)

For simplicity, we consider (34) and assume the solutions are,

$$X_1(x_1)=\sum_{r=0}^{\infty}a_nx_1^n$$

$$X_2(x_2)=\sum_{r=0}^{\infty}b_nx_2^n$$

And after a few steps, we can easily find out any one of the factorized solutions as displayed in Table 1.

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4. Analysis and applications

First of all, we investigate the solution to well-known Laplace's equation using different techniques as used in quantum calculus. As we said earlier, recently a number of physicists is using this plain-dealing quantum calculus in several fields to find out and understand the solutions in a better way. As a consequence, different techniques of $q-$calculus are seeming to be important to solve 1-D, 2-D, and 3-D differential equations as a basic theoretical requirement. Maximum of techniques, used in q-calculus are usually dealt with series solution method. Another attribute of quantum solution is, it reduces to the classical solution if and only if $q \to 1$. In addition, correctness of dimension of the q-solutions is another major issue in physics while mathematician do not take it as an urgent one. Therefore, in time of using new type of calculus in physics and engineering, dimensions of all the parameters should be taken care of.

In conventional calculus, factorized solution of 3-D, and 2-D Laplace's equations are represented in (9) and as mentioned in (11) and Table 1. Furthermore, small q-exponential $e_q(\pm kx) \to e^{kx}$, when $q \to 1$, and therefore, $sin_q(mx) \to \sin(mx), cos_q(mx) \to \cos(mx)\ and\ sinh_q(mx) \to \sinh(mx), cosh_q)(mx) \to \cosh(mx)$,when $q \to 1$. Therefore, all their corresponding quantum solutions (26), (31), and (32) will reduces to their classical form when $q \to 1$.

Secondly, the dimensions, of x_i $'s$ are basically [L], hence, the dimensions of all $k_i's$, $\alpha_i's$, and κ are $[L]^{-1}$. Hence all the exponentials, sine, cosine, sinh and cosh are dimensionless. Thus, there is no issue regarding the dimensions of the solutions.

CONCLUSION

In this paper, we have made a brief review about the developments of quantum calculus in diverse areas of physics, mathematics, and engineering. This work can be treated as a guidance to explore the applicability of quantum calculus in solving 2-D, 3-D differential equations often we see in physics and mathematics.

In this work, the quantum calculus is applied to solve 2-D, and 3-D Laplace's equation. The quantum solutions to the Laplace's equation have been described as they obtained from three different techniques. The validity of those solutions is also discussed by confirming the fact that those solution reduces to their classical form in a particular limiting situation q tends to 1, and then the accurateness of the dimensionality of the obtained solutions has been proved. Moreover, the main aim of this article is to show that, quantum calculus can be considered as a new pedagogical tool in introductory mathematical physics course.

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